

# GEOMETRIC STABILITY OF THE COULOMB ENERGY

ALMUT BURCHARD AND GREGORY R. CHAMBERS

ABSTRACT. The Coulomb energy of a charge that is uniformly distributed on some set is maximized (among sets of given volume) by balls. It is shown here that near-maximizers are close to balls.

## 1. INTRODUCTION AND MAIN RESULT

The *Coulomb energy* of a charge distribution  $f$  on  $\mathbb{R}^3$  is — up to a multiplicative physical constant — given by the singular integral

$$\mathcal{E}(f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|} dx dy.$$

According to the Riesz-Sobolev inequality, the energy of a positive charge distribution increases under symmetric decreasing rearrangement: If  $f^*$  is radially decreasing and equimeasurable with  $f$ , then

$$(1) \quad \mathcal{E}(f) \leq \mathcal{E}(f^*).$$

The physical reason is that symmetrization increases the interaction of the charges by reducing the typical distance between them. Equality holds only if the charge distribution is already radially decreasing about some point in  $\mathbb{R}^3$  [1]. Is this characterization of equality cases stable? If the two sides of Eq. (1) almost agree, how close must  $f$  be to a translate of  $f^*$ ?

We answer this question for charge distributions that are uniform on some set  $A \subset \mathbb{R}^3$  of finite volume. Let  $A^*$  be the ball of the same volume. With a slight abuse of notation, denote by

$$\mathcal{E}(A) = \int_A \int_A |x-y|^{-1} dx dy$$

the Coulomb energy of the uniform charge distribution on  $A$ .

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**Theorem 1.** *There exists a constant  $c > 0$  such that*

$$(2) \quad \frac{\mathcal{E}(A^*) - \mathcal{E}(A)}{\mathcal{E}(A^*)} \geq c \left( \inf_{\tau} \frac{\text{Vol}((\tau A) \triangle A^*)}{2\text{Vol}(A)} \right)^2.$$

for every  $A \subset \mathbb{R}^3$  of finite positive volume. Here,  $\tau$  runs over all translations in  $\mathbb{R}^3$ , and  $\triangle$  denotes the symmetric difference.

The exponent 2 is best possible; it is achieved for sets constructed from the unit ball by removing an annulus whose outer boundary is the unit sphere, and adding an annulus of the same volume whose inner boundary is the unit sphere.

Geometric stability results where a *deficit* (the deviation of a functional from its optimal value) controls some measure of *asymmetry* (the distance from the manifold of optimizers) have been established for many classical inequalities. The first results in that direction, due to Bonnesen in the 1920s, were quantitative improvements of the isoperimetric inequality for convex sets in the plane. Two papers from the early 1990s have inspired much recent progress. One is Hall's work on the isoperimetric inequality in  $\mathbb{R}^n$ , where he proves stability and raises the question of optimal exponents [2]; the other is the result of Bianchi and Egnell on the stability of the Sobolev inequality for  $\|\nabla f\|^2$  in dimension  $n \geq 3$  [3]. We refer the interested reader to the surveys [4, 5].

Less is known for non-local functionals that involve convolutions, even though stability results for those have important applications in Mathematical Physics [6]. In many variational problems for integral functionals, one can show by compactness arguments that all optimizing sequences must converge — modulo the symmetries of the functional — to extremals [7], but bounds for the asymmetry in terms of the deficit are a different matter. Very recently, Christ has introduced tools from additive number theory to prove stability of the Riesz-Sobolev inequality in one dimension [8]. Figalli and Jerison have obtained stability results on the Brunn-Minkowski inequality for non-convex sets in  $\mathbb{R}^n$  [9]. Fusco, Maggi, and Pratelli have proved stability of Talenti's inequality for the solutions of Poisson's equation [10, Theorem 2]. For the Coulomb energy, Guo conjectured that

$$(3) \quad \mathcal{E}(f^*) - \mathcal{E}(f) \geq c' \inf_{\tau} \mathcal{E}(f \circ \tau^{-1} - f^*)$$

with some constant  $c' > 0$ . (No normalization is required in this inequality, because both sides scale in the same way.) Since the Coulomb kernel is positive definite, the right hand side can be viewed as the square of a distance. The relationship between Eqs. (2) and (3) with  $f = \mathcal{X}_A$  will be clarified by Lemma 2.

The proof of Theorem 1 consists of two parts. After some preliminaries, we use the reflection positivity of the functional and a lemma of Fusco, Maggi, and Pratelli [11] to reduce the problem to sets that are symmetric under reflection at the coordinate hyperplanes. The second part of the proof requires an estimate for the Newton potential of symmetric sets. At the end of the paper, we briefly discuss stability for other Riesz kernels and in higher dimensions.

## 2. NOTATION, AND STABILITY IN HIGHER DIMENSIONS

By the *volume* of a set  $A \subset \mathbb{R}^n$ , denoted  $\text{Vol}(A)$ , we mean its  $n$ -dimensional Lebesgue measure. The centered open ball of the same volume is denoted by  $A^*$ ; its radius is called the *volume radius* of  $A$ , and denoted by  $R_A$ . The *Fraenkel asymmetry* of  $A$  is defined by

$$(4) \quad \alpha(A) = \inf_{\tau} \frac{\text{Vol}((\tau A) \triangle A^*)}{2\text{Vol}(A)}.$$

Further,  $B_R$  stands for the open ball of radius  $R$  centered at the origin, and  $\omega_n$  for the volume of the unit ball. The uniform surface measure that is induced on the sphere  $\partial B_r \subset \mathbb{R}^n$  by the ambient Lebesgue measure is denoted by  $\sigma$ .

We consider functionals of the form

$$(5) \quad \mathcal{E}(A) = \int_A \int_A |x - y|^{-\lambda} dx dy$$

with  $n \geq 3$  and  $\lambda \in [n - 2, n)$ . (The classical Coulomb energy corresponds to the case  $n = 3$  and  $\lambda = 1$ .) These functionals share the properties that they are reflection positive as well as positive definite (see [12]). Balls uniquely maximize them among sets of given volume [1]; balls are also the unique convex sets for which certain related overdetermined boundary-value problems have solutions [13]. By scaling,

$$(6) \quad \mathcal{E}(A) \leq \mathcal{E}(A^*) = \text{Constant} \cdot (\text{Vol}(A))^{2 - \frac{\lambda}{n}}.$$

The *deficit* of  $A$  is defined by

$$(7) \quad \delta(A) = \frac{\mathcal{E}(A^*) - \mathcal{E}(A)}{\mathcal{E}(A^*)}.$$

Each of the functionals can be expressed in terms of the corresponding *Riesz potential*

$$(8) \quad \Phi_A(x) = \int_A |x - y|^{-\lambda} dy, \quad x \in \mathbb{R}^n$$

as  $\mathcal{E}(A) = \int_A \Phi_A$ . By the Hardy-Littlewood-Sobolev inequality,  $\Phi_A$  lies in  $L^p$  for every  $p \geq n/\lambda$ . It is subharmonic on  $\mathbb{R}^n$  and smooth on the complement of  $A$ , though discontinuities may occur on  $\partial A$ . The Riesz potential is the unique solution of the pseudodifferential equation

$$(-\Delta)^{\frac{n-\lambda}{2}} \Phi = \text{Constant} \cdot \mathcal{X}_A$$

that decays at infinity. The constant  $c_{n,\lambda}$  can be computed with the help of the Fourier transform (see [14, Theorem 5.9]).

The Riesz-Sobolev inequality implies that

$$(9) \quad \int_E \Phi_A(x) dx \leq \int_{E^*} \Phi_{A^*}(x) dx$$

for every set  $E \subset \mathbb{R}^n$  of finite volume (see [14, Theorem 3.6]). In particular,  $\Phi_{A^*}$  is radially decreasing, and

$$(10) \quad \sup_x \Phi_A(x) \leq \Phi_{A^*}(0) = \int_{A^*} |y|^{-\lambda} dy = \frac{n\omega_n}{n-\lambda} R_A^{n-\lambda}.$$

Our proof of Theorem 1 fails in higher dimensions, because the crucial lower bound in Lemma 6 becomes negative. Nevertheless, we expect that the conclusion should hold — with the sharp exponent 2 and suitable constants  $c_{n,\lambda}$  — for the entire family of functionals in Eq. (5) with  $n \geq 1$  and positive  $\lambda \in [n-2, n)$ .

When  $n \geq 3$  and  $\lambda = n-2$ , we call  $\mathcal{E}(A)$  the *Coulomb energy* and  $\Phi_A$  the *Newton potential* associated with the uniform charge distribution on  $A$ . The Newton potential has many special properties related to Poisson's equation

$$-\Delta \Phi_A = n(n-2)\omega_n \mathcal{X}_A.$$

It is harmonic on the complement of  $A$ , subharmonic on  $\mathbb{R}^n$ , and satisfies the Gauss law. For later use, we compute the Newton potential of the centered ball of radius  $R$  as

$$(11) \quad \Phi_{B_R}(x) = \omega_n R^2 \cdot \begin{cases} \frac{n}{2} - \frac{n-2}{2} \left(\frac{|x|}{R}\right)^2, & |x| \leq R, \\ \left(\frac{|x|}{R}\right)^{-(n-2)}, & |x| \geq R, \end{cases}$$

and its Coulomb energy as

$$\mathcal{E}(B_R) = \frac{2n}{n+2} \omega_n^2 R^{n+2} = \frac{4}{n+2} \text{Vol}(B_R) \cdot \Phi_{B_R}(0).$$

A remarkable fact is *Talenti's comparison principle*, which says that the symmetric decreasing rearrangement of the Newton potential of a charge distribution is *pointwise* smaller than the potential resulting from symmetrizing the charge distribution itself [15],

$$(12) \quad (\Phi_A)^*(x) \leq \Phi_{A^*}(x), \quad x \in \mathbb{R}^n.$$

A similar inequality holds between the gradients of these functions. The inequalities are strict, unless  $A$  is essentially a ball [10, Theorem 1].

Eq. (12) is clearly stronger than the integrated version in Eq. (9). We will use Talenti's comparison principle to prove the following result.

**Theorem 2.** *Let  $\mathcal{E}$  be defined by Eq. (5) on  $\mathbb{R}^n$  with  $\lambda = n - 2$ . For each  $n \geq 3$ , there exists a constant  $c_n$  such that*

$$(13) \quad \frac{\mathcal{E}(A^*) - \mathcal{E}(A)}{\mathcal{E}(A^*)} \geq c_n \alpha(A)^{n+2}$$

for every  $A \subset \mathbb{R}^n$  of finite positive volume.

Note that the conclusion for  $n = 3$  is weaker than Theorem 1.

### 3. PRELIMINARY ESTIMATES

Throughout this section,  $A \subset \mathbb{R}^n$  is a set of finite positive volume, the functional  $\mathcal{E}(A)$  is given by Eq. (5) with  $\lambda \in [0, n)$ , and  $\Phi_A$  is the corresponding Riesz potential. We start by sharpening the bound on the maximum of  $\Phi_A$  from Eq. (10).

**Lemma 1.** *If  $A \subset \mathbb{R}^n$  has finite positive volume, then*

$$\sup_{x \in \mathbb{R}^n} \Phi_A(x) \leq \Phi_{A^*}(0) \cdot \left( 1 - \frac{\lambda(n-\lambda)}{n^2} \alpha(A)^2 \right).$$

*Proof.* By scaling, we may take  $A^*$  to be the unit ball. For  $x \in \mathbb{R}^n$ ,

$$\Phi_{A^*}(0) - \Phi_A(x) = \int_{A^* \setminus (x-A)} |y|^{-\lambda} dy - \int_{(x-A) \setminus A^*} |y|^{-\lambda} dy.$$

If  $\alpha(A) = \alpha$ , then each of the two regions of integration has volume at least  $\omega_n \alpha$ . The first integral is minimized when  $A^* \setminus (x-A)$  is an annulus whose outer boundary is the unit sphere, and the second integral is maximized when  $(x-A) \setminus A^*$  is an annulus whose inner boundary is the unit sphere. Using annuli of the appropriate volume, we calculate in polar coordinates

$$\begin{aligned} \Phi_{A^*}(0) - \Phi_A(x) &\geq n\omega_n \int_{(1-\alpha)^{1/n}}^1 r^{n-1-\lambda} dr - n\omega_n \int_1^{(1+\alpha)^{1/n}} r^{n-1-\lambda} dr \\ &= \frac{\lambda(n-\lambda)}{n^2} \Phi_{A^*}(0) \int_0^\alpha \int_{-s}^s (1+t)^{-1-\frac{\lambda}{n}} dt ds, \end{aligned}$$

where we have used the Fundamental Theorem of Calculus twice. By Jensen's inequality, the value of the double integral exceeds  $\alpha^2$ .  $\square$

Lemma 1 is needed for the proof of Theorem 2. In the next lemma, we use a similar estimate to relate  $\alpha(A)$  to the notion of asymmetry that appears Guo's conjecture, see Eq. (3). (It plays no role in the proofs of the main results.)

**Lemma 2.** *There exist positive constants  $c_{n,\lambda}$  and  $C_{n,\lambda}$  such that*

$$c_{n,\lambda}\alpha(A)^4 \leq \inf_{\tau} \frac{\mathcal{E}(\mathcal{X}_A \circ \tau^{-1} - \mathcal{X}_{A^*})}{\mathcal{E}(A^*)} \leq C_{n,\lambda}\alpha(A)^{2-\frac{\lambda}{n}}$$

for every  $A \subset \mathbb{R}^n$  of finite positive volume.

*Proof.* Assume by scaling that  $A^*$  is the unit ball, and set  $\alpha = \alpha(A)$ . For the first inequality, we translate  $A$  such that the infimum in the middle term is assumed when  $\tau$  is the identity. Since  $\mathcal{E}$  extends to a positive definite quadratic form on  $L^1 \cap L^\infty$ , we can use the Cauchy-Schwarz' inequality to obtain

$$\begin{aligned} \mathcal{E}(\mathcal{X}_A - \mathcal{X}_{A^*})^{\frac{1}{2}} \mathcal{E}(A^*)^{\frac{1}{2}} &\geq \int \int \frac{(\mathcal{X}_{A^*}(x) - \mathcal{X}_A(x)) \mathcal{X}_{A^*}(y)}{|x - y|^\lambda} dx dy \\ &= \int_{A^* \setminus A} \Phi_{A^*}(x) dx - \int_{A \setminus A^*} \Phi_{A^*}(x) dx \\ &\geq \int_{1-\alpha < |x|^n < 1} \Phi_{A^*}(x) dx - \int_{1 < |x|^n < 1+\alpha} \Phi_{A^*}(x) dx \\ &\geq \text{Constant} \cdot \alpha^2, \end{aligned}$$

where the constant depends on  $n$  and  $\lambda$ . We have used that  $\Phi_{A^*}$  is strictly radially decreasing to replace  $A^* \setminus A$  and  $A \setminus A^*$  with annuli. The last line follows since the gradient of  $\Phi_{A^*}$  vanishes only at  $x = 0$ .

For the second inequality, we translate  $A$  so that the infimum in Eq. (4) is assumed at the identity. The Hardy-Littlewood-Sobolev inequality implies that

$$\inf_{\tau} \mathcal{E}(\mathcal{X}_A \circ \tau^{-1} - \mathcal{X}_{A^*}) \leq C_{n,\lambda} \|\mathcal{X}_A - \mathcal{X}_{A^*}\|_{\frac{2n}{2n-\lambda}}^2 = C_{n,\lambda} \alpha^{2-\frac{\lambda}{n}}. \quad \square$$

We need a few more lemmas for the proof of Theorem 1. The following integral representation will appear several times.

**Lemma 3.** *Let  $\rho(r)$  denote the volume radius of  $A \cap B_r$ . For any  $R > 0$ ,*

$$\mathcal{E}(A^*) - \mathcal{E}(A) \geq 2 \int_R^\infty \int_{A \cap \partial B_r} \left( \Phi_{(A \cap B_r)^*} \Big|_{\partial B_{\rho(r)}} - \Phi_{A \cap B_r}(x) \right) d\sigma(x) dr.$$

*Proof.* The functional can be written as

$$\begin{aligned}
 \mathcal{E}(A) &= 2 \int_A \int_A \mathcal{X}_{\{|x| > |y|\}} |x - y|^{-\lambda} dy dx \\
 (14) \quad &= 2 \int_{A \cap B_R} \Phi_{A \cap B_{|x|}}(x) dx + 2 \int_{A \setminus B_R} \Phi_{A \cap B_{|x|}}(x) dx \\
 &= \mathcal{E}(A \cap B_R) + 2 \int_R^\infty \int_{A \cap \partial B_r} \Phi_{A \cap B_r}(x) d\sigma(x) dr.
 \end{aligned}$$

Applying Eq. (14) to  $A^*$  with  $\rho(R)$  in place of  $R$ , we see that

$$\begin{aligned}
 \mathcal{E}(A^*) &= \mathcal{E}(B_{\rho(R)}) + 2 \int_{\rho(R)}^\infty \Phi_{B_\rho} \Big|_{|x|=\rho} n\omega_n \rho^{n-1} d\rho \\
 &= \mathcal{E}((A \cap B_R)^*) + 2 \int_R^\infty \Phi_{(A \cap B_r)^*} \Big|_{|x|=\rho(r)} \sigma(A \cap \partial B_r) dr.
 \end{aligned}$$

In the first line, we have used that  $B_{\rho(R)} \subset A^*$ . The Jacobian for the change of variables in the next step is determined by the relation  $n\omega_n \rho^{n-1} d\rho = \sigma(A \cap \partial B_r) dr$ . Since  $\mathcal{E}(A \cap B_R) \leq \mathcal{E}((A \cap B_R)^*)$  by Eq. (1), the claim follows upon subtracting Eq. (14).  $\square$

The next lemma reduces the stability problem to bounded sets.

**Lemma 4.** *For every  $n \geq 3$  and  $\lambda \in [n-2, n)$  there are positive constants  $\alpha_{n,\lambda}$  and  $c_{n,\lambda}$  with the following property. Given a set  $A \subset \mathbb{R}^n$  of finite positive volume with  $\alpha_0 := \text{Vol}(A \triangle A^*) / (2\text{Vol}(A)) \leq \alpha_{n,\lambda}$ , there exists a set  $\tilde{A}$  of the same volume such that*

$$\tilde{A} \subset (1 + c_{n,\lambda} \alpha_0^{1-\frac{\lambda}{n}}) A^*, \quad \frac{\text{Vol}(\tilde{A} \triangle A^*)}{2\text{Vol}(\tilde{A})} = \alpha_0, \quad \delta(\tilde{A}) \leq \delta(A).$$

*If  $A$  is symmetric about the origin, then so is  $\tilde{A}$ .*

*Proof.* By scaling, we may assume that  $A^*$  is the unit ball, i.e.,  $R_A = 1$ . Given  $R > (1 + \alpha_0)^{1/n}$ , determine  $r \in (1, R)$  such that

$$\tilde{A} = (A \cap B_R) \cup (B_r \setminus A^*)$$

has the same volume as  $A$ . By construction,  $\text{Vol}(\tilde{A} \triangle A^*) = \text{Vol}(A \triangle A^*)$ , and  $r \leq (1 + \alpha_0)^{1/n}$ .

We want to choose  $R$  so that  $\mathcal{E}(\tilde{A}) \geq \mathcal{E}(A)$ . It follows from Eq. (14) that

$$\mathcal{E}(A) \leq \mathcal{E}(A \cap B_R) + 2\text{Vol}(A \setminus B_R) \cdot \sup_{|x| \geq R} \Phi_A(x).$$

Since  $\Phi_A \leq \Phi_{A^*} + \Phi_{A \setminus A^*}$ , Eq. (10) implies

$$\Phi_A(x) \leq \Phi_{A^*}(x) + \frac{n\omega_n}{n-\lambda} \alpha_0^{1-\lambda/n}.$$

Similarly, since  $\tilde{A} \cap A = A \cap B_R$  by construction,

$$\begin{aligned} \mathcal{E}(\tilde{A}) &= \int_{\tilde{A} \cap A} \Phi_{\tilde{A} \cap A}(x) dx + \int_{\tilde{A} \setminus A} 2\Phi_{\tilde{A} \cap A}(x) + \Phi_{\tilde{A} \setminus A}(x) dx \\ &\geq \mathcal{E}(A \cap B_R) + 2\text{Vol}(A \setminus B_R) \cdot \inf_{|x| \leq r} \Phi_{\tilde{A}}(x) - \mathcal{E}(\tilde{A} \setminus A), \end{aligned}$$

and

$$\Phi_{\tilde{A}}(x) \geq \Phi_{A^*}(x) - \frac{n\omega_n}{n-\lambda} \alpha_0^{1-\lambda/n}.$$

We use Eq. (6) and the fact that  $\text{Vol}(\tilde{A} \setminus A) = \text{Vol}(A \setminus B_R) \leq \omega_n \alpha_0$  to estimate

$$\mathcal{E}(\tilde{A} \setminus A) \leq \text{Constant} \cdot \text{Vol}(A \setminus B_R) \alpha_0^{1-\frac{\lambda}{n}}.$$

Collecting terms, we obtain that

$$\mathcal{E}(\tilde{A}) - \mathcal{E}(A) \geq 2\text{Vol}(A \setminus B_R) \cdot \left( \Phi_{A^*} \Big|_{|x|=R}^{|x|=(1+\alpha_0)^{1/n}} - \text{Constant} \cdot \alpha_0^{1-\frac{\lambda}{n}} \right).$$

We have used that  $\Phi_{A^*}$  is radially decreasing to replace the inner radius  $r$  by  $(1 + \alpha_0)^{1/n}$ . Since  $\Phi_{A^*}$  is a smooth, strictly radially decreasing function whose gradient does not vanish outside  $A^*$ , there exists a constant  $c_{n,\lambda}$  such that the right hand side is positive for

$$R = 1 + c_{n,\lambda} \alpha_0^{1-\lambda/n}$$

when  $\alpha_0$  is sufficiently small.  $\square$

We now introduce reflection symmetries to the problem. The basic construction is as follows. Given a hyperplane that bisects  $A$  into two parts of equal volume, the set  $A$  is replaced by the union of one of these parts with its mirror image. We refer to the two sets that can be obtained in this way as *symmetrizations* of  $A$  at the hyperplane. Clearly, the symmetrizations have the same volume as  $A$ .

**FMP Symmetrization Lemma** [11, Theorem 2.1]. *For every  $n \geq 1$  there is a positive constant  $c_n$  with the following property. Given a set  $A \subset \mathbb{R}^n$  of finite positive volume, there exists a set  $\tilde{A}$  obtained by successive symmetrization of  $A$  at  $n$  orthogonal hyperplanes such that*

$$\alpha(\tilde{A}) \geq c_n \alpha(A).$$

**Lemma 5.** *If  $\lambda \in [n-2, n)$ , then the set constructed in the FMP lemma satisfies*

$$\delta(\tilde{A}) \leq 2^n \delta(A).$$



*Proof.* Consider the two possible symmetrizations  $A_+$  and  $A_-$  of  $A$  at a single hyperplane. Since  $\lambda \in [n-2, n)$ , the functional is reflection positive, meaning that

$$\mathcal{E}(A_+) + \mathcal{E}(A_-) \geq 2\mathcal{E}(A),$$

see [12, Section 1.1]. Using that  $(A_+)^* = (A_-)^* = A^*$ , we subtract both sides of the inequality from  $\mathcal{E}(A^*)$  to obtain

$$\delta(A_+) + \delta(A_-) \leq 2\delta(A),$$

and conclude that the symmetrized sets satisfy  $\delta(A_\pm) \leq 2\delta(A)$ . The claim follows by repeating the construction  $n$  times.  $\square$

We translate and rotate  $\tilde{A}$  to a set that is symmetric at the coordinate hyperplanes, and thus symmetric under  $x \mapsto -x$ . Such sets have the useful property that their asymmetry is comparable to their symmetric difference from a *centered* ball [11, Lemma 2.2]. The following estimate for the potential is the key to the proof of Theorem 1.

**Lemma 6.** *If  $A \subset B_r$  is symmetric about the origin, then*

$$\Phi_A(x) \leq \Phi_{B_r}(x) - (\sqrt{2}r)^{-\lambda} \text{Vol}(B_r \setminus A)$$

for all  $x \in \partial B_r$ .

*Proof.* Let  $x \in \partial B_r$  be given. The function

$$f(y) = \frac{1}{2} (|y - x|^{-\lambda} + |y + x|^{-\lambda})$$

assumes its minimum at a point on  $\partial B_r$  equidistant to  $x$  and  $-x$ , and the minimum value is  $(\sqrt{2}r)^{-\lambda}$ . Since  $A$  is symmetric,

$$\Phi_A(x) = \int_A |x - y|^{-\lambda} dy \geq (\sqrt{2}r)^{-\lambda} \text{Vol}(A).$$

The claim follows by replacing  $A$  with  $B_r \setminus A$ .  $\square$

For the Newton potential of  $A \subset B_r$ , Lemma 6 implies that

$$\begin{aligned} \Phi_{A^*} \Big|_{\partial A^*} - \sup \Phi_A \Big|_{\partial B_r} &\geq \omega_n \left( R_A^2 - r^2 + \frac{r^n - R_A^n}{(\sqrt{2}r)^{n-2}} \right) \\ (15) \qquad \qquad \qquad &= \omega_n R_A^2 (-2 + n2^{1-\frac{n}{2}}) \left( \frac{r}{R_A} - 1 \right) + O\left( \frac{r}{R_A} - 1 \right)^2 \end{aligned}$$

uniformly in  $A$  as  $\frac{r}{R_A} \rightarrow 1$ . Note that the leading term is positive in dimension  $n = 3$ .

## 4. PROOF OF THE MAIN RESULTS

*Proof of Theorem 1.* We specialize to the case of the Coulomb energy in  $\mathbb{R}^3$ , where  $\lambda = 1$ . We want to find a constant  $c > 0$  such that  $\delta(A) \geq c\alpha(A)^2$  for all sets of finite positive volume  $A \subset \mathbb{R}^3$ . By scaling, we may assume that  $\text{Vol}(A) = \omega_3 = 4\pi/3$ , so that  $A^*$  is the unit ball. Since  $\alpha(A) \leq 1$  by definition, it suffices to prove the claim for  $\alpha$  sufficiently small.

By Lemma 5 we may assume that  $A$  is symmetric about the origin. Therefore, by [11, Lemma 2.2],

$$\alpha_0 := \text{Vol}(A \triangle A^*)/(2\omega_3) \leq 3\alpha(A).$$

By Lemma 4 we may assume furthermore that  $A$  lies in the ball of radius

$$R_0 = 1 + c_{3,1}\alpha_0^{\frac{2}{3}},$$

provided that  $3\alpha(A) \leq \alpha_{3,1}$ . We use Lemma 3 with  $R = 1$  to see that

$$\mathcal{E}(A^*) - \mathcal{E}(A) \geq 2 \int_1^{R_0} \int_{A \cap \partial B_r} \Phi_{(A \cap B_r)^*} \Big|_{\partial B_{\rho(r)}} - \Phi_{A \cap B_r}(x) d\sigma(x) dr,$$

where  $\rho(r)$  is the volume radius of  $A \cap B_r$ . By Eq. (15), the integrand is bounded from below by

$$\Phi_{(A \cap B_r)^*} \Big|_{\partial B_{\rho(r)}} - \Phi_{A \cap B_r}(x) \geq \omega_3 \inf_{1 \leq r \leq R_0} \left\{ \rho(r)^2 - r^2 + \frac{r^3 - \rho(r)^3}{\sqrt{2}r} \right\}.$$

The function inside the braces can be written as a product

$$(r^3 - \rho(r)^3) \left( -\frac{r + \rho(r)}{r^2 + r\rho(r) + \rho(r)^2} + \frac{1}{\sqrt{2}r} \right).$$

Since the first factor is non-decreasing in  $r$ , it is bounded from below by  $1 - \rho(1)^3 = \alpha_0$ . This gives for the integral

$$\mathcal{E}(A^*) - \mathcal{E}(A) \geq \frac{8\pi}{3} \alpha_0^2 \cdot \inf_{(1-\alpha_0)^{1/3} \leq \rho \leq r \leq R_0} \left\{ -\frac{r + \rho}{r^2 + r\rho + \rho^2} + \frac{1}{\sqrt{2}r} \right\}.$$

The infimum is strictly positive for  $\alpha_0$  sufficiently small. Since  $\alpha_0 \geq \alpha(A)$  by definition, the theorem follows.  $\square$

The proof of Theorem 1 used that the Coulomb kernel  $|x|^{-1}$  is symmetric decreasing and reflection positive, without taking advantage of the special properties of the Newton potential. Since all estimates used in the proof depend continuously on  $\lambda$ , the conclusion extends to nearby values.

**Corollary** *Let  $\mathcal{E}_\lambda$  be defined by Eq. (5) for  $n = 3$  and  $\lambda > 1$ , and let  $\delta_\lambda$  be the corresponding deficit given by Eq. (7). For every  $\lambda$  sufficiently close to 1 there exists a constant  $c_\lambda$  such that*

$$\delta_\lambda(A) \geq c_\lambda \alpha(A)^2$$

for all  $A \subset \mathbb{R}^3$ .

Finally we turn to the Coulomb energy in dimension  $n \geq 3$ .

*Proof of Theorem 2.* Let  $n \geq 3$  and  $\lambda = n - 2$ . Assume, by scaling, that  $A^*$  is the unit ball, and let  $\alpha = \alpha(A)$  be the asymmetry of  $A$ . Since  $\int_A \Phi_A \leq \int_{A^*} (\Phi_A)^*$ ,

$$\mathcal{E}(A^*) - \mathcal{E}(A) \geq \int_{A^*} \Phi_{A^*} - (\Phi_A)^* dx.$$

By Talenti's comparison principle, the integrand is nonnegative. Moreover, by Lemma 1 and Eq. (11),

$$\begin{aligned} \Phi_{A^*}(x) - (\Phi_A)^*(x) &\geq \Phi_{A^*}(x) - \sup_y \Phi_A(y) \\ &\geq \frac{n-2}{2} \omega_n \left( \frac{2\alpha^2}{n} - |x|^2 \right)_+. \end{aligned}$$

Integration yields

$$\begin{aligned} \mathcal{E}(A^*) - \mathcal{E}(A) &\geq \frac{n-2}{2} \omega_n \int_{A^*} \left[ \frac{2\alpha^2}{n} - |x|^2 \right]_+ dx \\ &= \frac{n-2}{2n} \mathcal{E}(A^*) \cdot \left( \frac{\sqrt{2}\alpha}{\sqrt{n}} \right)^{n+2}, \end{aligned}$$

which proves Eq. (13) with  $c_n = (n-2)2^{n/2}/n^{2+n/2}$ .  $\square$

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## REFERENCES

- [1] E.H. Lieb. Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Studies in Applied Mathematics* **57**:93–105 (1977)
- [2] R.R. Hall. A quantitative isoperimetric inequality in  $n$ -dimensional space. *Journal für die Reine und Angewandte Mathematik* **428**:161–176 (1992)
- [3] G. Bianchi and H. Egnell. A note on the Sobolev inequality. *Journal of Functional Analysis* **100**:18–24 (1991)
- [4] R. Osserman. Bonnesen-style isoperimetric inequalities. *The American Mathematical Monthly* **86**:1–29 (1979)
- [5] F Maggi. Some methods for studying stability in isoperimetric type problems. *Bulletin of the American Mathematical Society* **45**:367–408 (2008)
- [6] E.A. Carlen. Stability results for sharp inequalities and their application in mathematical physics: An introduction to the critical droplet problem as a case study. *Lecture notes* (2011)
- [7] A. Burchard and Y. Guo. Compactness via symmetrization. *Journal of Functional Analysis* **214**:40–73 (2004)
- [8] M. Christ. Near equality in the Riesz-Sobolev inequality. *Preprint arXiv:1309.5856* (2013)
- [9] A. Figalli and D. Jerison. Quantitative stability for sumsets in  $\mathbf{R}^n$ . *Journal of the European Mathematical Society* **17**:1079–1106 (2015)
- [10] N. Fusco, F. Maggi, and A. Pratelli. Personal communication (unpublished manuscript, 20 pages, 2009)
- [11] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Annals of Mathematics* (2) **168**:941–980 (2008)
- [12] R.L. Frank and E.H. Lieb. Inversion positivity and the sharp Hardy-Littlewood-Sobolev inequality. *Calculus of Variations and Partial Differential Equations* **39**:85–99 (2010)
- [13] W. Reichel. Characterization of balls by Riesz-potentials. *Annali di Matematica Pura ed Applicata* **188**:235–245 (2009)
- [14] E.H. Lieb and M. Loss. *Analysis*. Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, RI, first/second edition (1997/2001).
- [15] G. Talenti. Elliptic equations and rearrangements. *Annali della Scuola Normale Superiore di Pisa — Classe di Scienze* **3**:697–718 (1976)

A. BURCHARD, UNIVERSITY OF TORONTO, DEPARTMENT OF MATHEMATICS,  
40 ST. GEORGE STREET, ROOM 6290, TORONTO, CANADA M5S 2E4  
almut@math.toronto.edu

G.R. CHAMBERS, UNIVERSITY OF CHICAGO, DEPARTMENT OF MATHEMATICS,  
5734 S. UNIVERSITY AVENUE, ROOM 208 C, CHICAGO, IL 60637  
chambers@math.uchicago.edu